Indian Statistical Institute, Bangalore
B. Math (Hons.) Third Year
First Semester - Combinatorics and graph theory

Semestral Exam Date: November 22, 2018
Maximum marks: 100 Duration: 3 hours

Instruction: There are six questions. Answer any five.

1. Let G be a finite graph such that any two distinct vertices of G have a unique common neighbour.
   (a) Show that any two non-adjacent vertices of G have the same degree.
   (b) If the complement of G is disconnected then show that there is a vertex of G which is adjacent in G with all other vertices. In this case determine G up to isomorphism.
   (c) Conclude that if no vertex of G is adjacent to all the other vertices then G is regular, say of degree K, and G has exactly $K^2 - K + 1$ vertices. [10 + 4 + 6 = 20]

2. Let G be a finite connected regular graph with $\vartheta > 1$ vertices. Then show that
   (a) any coclique of G has size at most $\vartheta/2$ and
   (b) G has zero or two cocliques of size $\vartheta/2$.
   (Hint: if C is a coclique of G then count in two ways the edges of G joining a vertex of C to a vertex outside C.) [10 + 10 = 20]

3. Let D be a $3-(22,6,1)$ design. Fix a point x of D and let X be the set of all blocks of D not containing x. Define the binary relation I on X as follows. For $B_1, B_2 \in X, B_1\ I\ B_2$ iff either $B_1 = B_2$ or $B_1 \cap B_2 = \varphi$. Then show that the incidence system $E = (X, X, I)$ is square 2-design. Compute the parameters of E. [10 + 10 = 20]

4. Let G be a regular bipertite graph of degree k. Suppose there is a constant $\lambda > 0$ such that any two vertices of G from the same part have exactly λ common neighbours. Show that
   (a) If the total number of vertices of G is a multiple of 4 then $k - \lambda$ must be a perfect square.
   (b) Give an example to show that if the number of vertices of G is not a multiple of 4 then the conclusion of (a) may be false. [10 + 10 = 20]

5. If q is a prime power then prove that there is a $3-(q^2+1, q+1, 1)$ design. Show that all the one-point contractions of the design constructed by you are isomorphic to $EG(2, q)$. [10 + 10 = 20]

6. let D be a $t-(v, k, \lambda)$ design such that $v = 2k + 1$ and t is even. Then show that D has an one-point extension. [20]